MOTION OF A BUBBLE IN A VISCOUS LIQUID

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The discussion concerns steady-state flow of a viscous fluid around a spherical bubble at small Reynolds number R. Asymptotic matching [1] provides a way of calculating the resistance force, which agrees well with the measured force for R < 5. The rate of growth or dissolution of the bubble is calculated on the assumption that the Péclet number is large.

It follows from a survey [2] of the rise rates of single bubbles in liquids that the viscous resistance for $R \ll 1$ (R is Reynolds number) coincides with the Stokes force for a hard sphere of the same radius. Levich [3] considers that the cause of this is the adsorption of surfactants, which produce an immobile film at the surface. The Stokes force is replaced by the Adamar-Rybchinski one [3] for a spherical bubble with a free surface as R increases, and experiment [2] shows that the transition point is dependent on the properties of the liquid over a fairly wide range in R (about 10^{-4} to 20).

It is assumed here that there is flow around a spherical bubble with a free surface for $R \ll 1$, with the equation [1]

$$D^4\psi = \frac{R}{r^2 \sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} + 2\operatorname{ctg}\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\theta}\right)D^2\psi \quad (1)$$

$$\left(D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}, \quad R = \frac{ua}{\nu}\right)$$

Here r is the distance from the center of the bubble as divided by the bubble radius a; θ is the polar angle (reckoned from the direction of **u**, the velocity vector for the incident flow); and ν is the kinematic viscosity.

The boundary conditions state that v_r (radial component of the velocity) becomes zero at the surface of the bubble, as do $\sigma_{r\theta}$ (the components of the stress tensor). Also, the incident flow is uniform, and so

$$\frac{\partial\psi}{\partial\theta} = 0, \quad \frac{\partial^2\psi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial r^2} + 2\frac{\partial\psi}{\partial r} = 0 \quad \text{for} \quad r = 1 \quad (2)$$

$$\psi \to {}^1/_2 r^2 \sin^2\theta \quad \text{for} \quad r \to \infty$$

We seek a solution in the form

$$\psi = \psi_0 + R\psi_1 + R^2\psi_2 + \ldots \qquad (R \ll 1)$$

Then it follows from (1) and (2) that

$$\psi_0 = {}^1/_2\, r(r-1)\sin^2\theta \quad (3)$$

$$D^4\psi_1 = -3r^{-2}(1 - r^{-1})\sin^2\theta\cos\theta \quad (4)$$

A particular solution to (4) is

$$\psi_1 = -[{}^1/_8 r(r-1) + a_5 r^5 + a_3 r^3 + a_0 + a_{-2} r^{-2}]\sin^2\theta\cos\theta$$

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The principle of minimal singularity [1] implies $a_5 = a_3 = 0$; in fact, it is impossible to link this solution to the exterior expansion of the current function if we retain these terms.

Then the following is the general solution to (4) that satisfies the boundary conditions at $r = 1$ and $r \to \infty$ while increasing not more rapidly than as r^2:

$$\psi_1 = (A - {}^1/_8 \cos\theta)\, r\, (r - 1) \sin^2\theta \tag{5}$$

Here A is a constant to be determined from the linking to the exterior expansion.

Further, we get

$$D^4\psi_2 = P_1(r)Q_1(\theta) + P_2(r)Q_2(\theta) + P_3(r)Q_3(\theta) \tag{6}$$

$$\begin{pmatrix} Q_1(\theta) = \sin^2\theta, & Q_2(\theta) = \sin^2\theta\cos\theta \\ Q_3(\theta) = (5\cos^2\theta - 1)\sin^2\theta, & P_1(r) = {}^1/_5 r^{-1}(1 - {}^3/_2 r^{-1}) \end{pmatrix}$$

The Q_i are eigenfunctions of the operator D^2.

We seek a solution to (6) as

$$\psi_2 = f_1(r)Q_1(\theta) + f_2(r)Q_2(\theta) + f_3(r)Q_3(\theta)$$

We restrict our definition of $f_1(r)$ to

$$f_1(r) = a_3 r^3 + a_2 r^2 + a_1 r + b_2 r^2 \ln r \tag{7}$$

Substitution of (7) into (6) gives the coefficients, with $b_2 = {}^1/_{20}$.

We see from (5) and (7) that the corrections to the current function do not satisfy the boundary condition at infinity, because [1] the ratio of the convective terms to the viscous ones is of the order of Rr for $r \to \infty$. Although this ratio is small for $r \sim 1$, the convective terms in (1) cannot be considered as a small correction to the Stokes equation for r large.

The Oseen equations take some account of the convective terms and correctly describe the velocity distribution at large distances [1]:

$$\left(D^2 - \cos\theta\frac{\partial}{\partial\rho} + \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right) D^2\Psi = 0 \tag{8}$$

$$\left(D^2 = \frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}, \quad \rho = Rr, \quad \Psi(\rho, \theta) = \psi\left(\frac{\rho}{R}, \theta\right)\right)$$

The following is a solution to (8) that satisfies the conditions at infinity and the principle of minimum singularity at the origin:

$$\Psi = \frac{\rho^2}{2R^2}\sin^2\theta - \frac{B}{R}(1 + \cos\theta)\left\{1 - \exp\left[-\frac{\rho}{2}(1 - \cos\theta)\right]\right\} \tag{9}$$

Here B is a constant to be determined by asymptotic linking. The following is (9) rewritten in terms of interior variables:

$$\Psi = \frac{r^2}{2}\sin^2\theta - \frac{B}{R}(1 + \cos\theta)\left\{1 - \exp\left[-\frac{Rr}{2}(1 - \cos\theta)\right]\right\}$$

and for R small becomes

$$\Psi = {}^1/_2\, r^2 \sin^2\theta - {}^1/_2\, Br \sin^2\theta$$

which should coincide with the one-term interior expansion of (3), so B = 1.

The two-term interior expansion of (3) and (5) is put as follows in terms of interior variables:

$$\psi = {}^1/_2 (\rho / R)(\rho / R - 1)(1 + AR - {}^1/_4 R\cos\theta)\sin^2\theta$$

and for R small becomes

$$\psi = {}^1/_2\,(\rho\,/\,R)^2\,(1 - R\,/\,\rho + AR - {}^1/_4 R\cos\theta)\sin^2\theta$$

which should coincide with (9) as written up to terms of the order of R inclusive:

$$\Psi = {}^1/_2\, r^2 \sin^2\theta - {}^1/_2 r \sin^2\theta + {}^1/_8 r^2 R\,(1 - \cos\theta)\cdot\sin^2\theta$$

This means that $A = {}^1/_4$.

Then the following is the current function near the bubble surface up to terms of the order of R:

$$\psi = {}^1/_2 r\,(r - 1)\,[1 + {}^1/_4\, R\,(1 - \cos\theta)]\sin^2\theta \tag{10}$$

If ψ_2 (the next correction to the current function) is written in terms of exterior variables, the other terms will include a higher-order term:

$$\psi_2 = -\frac{\rho^2 \ln R}{20 R^2}\sin^2\theta \tag{11}$$

If we seek a solution far from the sphere as a power series in R, with the Oseen solution as the zeroth approximation, the next approximation will not contain terms that link up with the function of (11) [4], so we have to eliminate this term by adding to the interior expansion a term of the form

$$\psi = \psi_0 + R\psi_1 + {}^1/_{20} R^2 \ln R\; r\,(r - 1)\sin^2\theta + \ldots$$

The current function near the bubble is then as follows up to terms of the order of $R^2 \ln R$:

$$\psi = {}^1/_2 r\,(r - 1)\,[1 + {}^1/_4\, R\,(1 - \cos\theta) + {}^1/_{10}\, R^2 \ln R]\sin^2\theta$$

We get the following as the force on a bubble moving steadily in a liquid:

$$F = 4\pi\mu a u\,(1 + {}^1/_4 R + {}^1/_{10} R^2 \ln R) \tag{12}$$

Here $\mu = \rho'\nu$ is the dynamic viscosity of the liquid.

If we use only the first two terms in (12), the following is the R dependence of the resistance coefficient c_D:

$$c_D = \frac{2F}{\pi\rho' a^2 u^2} = \frac{8}{R}\left(1 + \frac{1}{4}R\right) \tag{13}$$

This agrees well with experiment [2] up to $R \approx 5$, as Fig. 1 shows, where curve 1 is the Stokes solution for a solid sphere, 2 is the Stokes solution with the Oseen correction, 3 is the Adamar-Rybchinski solution for a sphere with a free surface, and 4 is the result from (13). The experimental curves are represented by the points.

Although the Stokes formula also agrees with experiment for $1.5 < R < 5$, formula (13) agrees with experiment throughout the region $R < 5$, so we may assume that flow around a bubble may be considered as flow around a sphere with a free surface rather than as flow around a solid sphere. The difference between the two flow conditions is unimportant in determining the rate of rise for $1.5 < R < 5$ but is important in calculating the rate of growth or shinkage by diffusion.

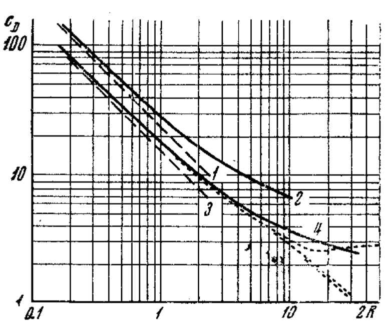

Fig. 1

In the Stokes condition, when $ua \gg D$ ($\nu \gg D$), the diffusion flux to the surface of a solid sphere is [3]

$$I = 7.98\,(uD^2a^4)^{1/3}\,(c_\infty - c_a) \tag{14}$$

Here C_∞ is the gas concentration far from the bubble, while c_a is the concentration at the surface.

A similar calculation via (10) gives the following result for flow around a sphere with a free surface:

$$I = 5.79\ (uDa^3)^{1/2}\,(1 + {}^1/_4 R)^{1/2}\,(c_\infty - c_a) \tag{15}$$

This becomes Levich's formula [3] for R→ 0. The correction to the first term in the R expansion is about 50% for R=5.

We can substitute the rate of steady-state rise of a bubble into (15):

$$u = \frac{1}{3}\frac{ga^2}{\nu}\frac{1}{1+{}^1/_4 R} \tag{16}$$

Here g is the acceleration due to gravity.

Then the total flux to the surface of the bubble is

$$I = 3.35\ (Da^5 g\ /\ \nu)^{1/2}\,(c_\infty - c_a) \tag{17}$$

The factor dependent on R in (17) cancels out also when we incorporate the next term in the expansion of the current function as a series in R.

Equation (17) also describes the rate of growth or shrinkage of a spherical bubble by diffusion for $\sqrt{R} \gg 1$ in the absence of surfactants [3], so it is suitable for the entire range where there is no flow detachment.

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